

Canonical Transformations and Hamiltonian Evolutionary Systems

Samer Ashhab

Department of Mathematics, University of New Orleans, New Orleans,
LA 70148.

E-mail: salashha@uno.edu

Abstract

In many Lagrangian field theories one has a Poisson bracket defined on the space of local functionals. We find necessary and sufficient conditions for a transformation on the space of local functionals to be canonical in three different cases. These three cases depend on the specific dimensions of the vector bundle of the theory and the associated Hamiltonian differential operator. We also show how a canonical transformation transforms a Hamiltonian evolutionary system and its conservation laws. Finally we illustrate these ideas with three examples.

Keywords: Hamiltonian evolutionary systems, Poisson bracket, jet bundle.

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1 Introduction

In Hamiltonian evolutionary systems one has a Poisson bracket. This Poisson bracket is defined on the space of local functionals of a fiber bundle. The system is characterized by a Hamiltonian which is a local functional and the associated Poisson bracket. The Poisson bracket in turn is characterized by a *Hamiltonian* differential operator (see the discussion in the next section). The solutions to the evolutionary system are sections of the fiber bundle. Generally an evolutionary system has the form

$$u_t = \mathcal{D}\delta\mathcal{H}$$

where \mathcal{D} is the associated *Hamiltonian* differential operator, δ the variational derivative and \mathcal{H} the Hamiltonian of the system. The variational derivative of a local functional $\mathcal{P} = \int_M P\nu$ has the same components as $\mathbf{E}(P\nu)$ where \mathbf{E} is the Euler-Lagrange operator and P is a local function representing the local functional \mathcal{P} , while ν is a volume element on the base manifold M . A good reference on evolutionary systems and their theory can be found in [5] and [8] for example. The reader may consult [6] as well.

An action on the bundle induces an action/transformation on the space of local functionals. If this transformation preserves the Poisson bracket we call it a canonical transformation. This terminology was used in [1] and is in analogy to the one used in [7] for the case of symplectic manifolds. In [1] canonical transformations were studied in the case the Poisson bracket is defined by a differential operator of order *zero*. In this paper we study canonical transformations in the case of higher order differential operators. We work with vector bundles with m -dimensional fibers and n -dimensional base space. We consider special cases of higher order differential operators which we find applications for and intend to broaden our study in future work.

A canonical transformation will transform the evolutionary system, its Hamiltonian, and its conservation laws where a new system is obtained. We will show that the conditions for a canonical transformation are rather restrictive in some cases. We illustrate by a few examples at the end of the paper.

It would be interesting to study the sh-Lie algebra structure and reduction as was done in [1] and [3]. However we leave this for a possible future work.

2 Background material

Let $\pi : E \rightarrow M$ be a vector bundle of dimension $m+n$ and with a base space an n -dimensional manifold M . Let $J^\infty E$ be the infinite jet bundle of E . The restriction of the infinite jet bundle over an appropriate open set $U \subset M$ is trivial with fiber an infinite dimensional vector space V^∞ . The bundle

$$\pi^\infty : J^\infty E_U = U \times V^\infty \rightarrow U$$

then has induced coordinates given by

$$(x^i, u^a, u_i^a, u_{i_1 i_2}^a, \dots).$$

We use multi-index notation and the summation convention throughout the paper. If $j^\infty \phi$ is the section of $J^\infty E$ induced by a section ϕ of the bundle E , then $u^a \circ j^\infty \phi = u^a \circ \phi$ and

$$u_I^a \circ j^\infty \phi = (\partial_{i_1} \partial_{i_2} \dots \partial_{i_r})(u^a \circ j^\infty \phi)$$

where r is the order of the symmetric multi-index $I = \{i_1, i_2, \dots, i_r\}$, with the convention that, for $r = 0$, there are no derivatives. For more details see [2] and [6].

Let Loc_E denote the algebra of local functions where a local function on $J^\infty E$ is defined to be the pull-back of a smooth function on some finite jet bundle $J^p E$ via the projection from $J^\infty E$ to $J^p E$. Let Loc_E^0 denote the subalgebra of Loc_E such that $P \in Loc_E^0$ iff $(j^\infty \phi)^* P$ has compact support for all $\phi \in \Gamma E$ with compact support and where ΓE denotes the set of sections of the bundle $E \rightarrow M$. The de Rham complex of differential forms $\Omega^*(J^\infty E, d)$ on $J^\infty E$ possesses a differential ideal, the ideal C of contact forms θ which satisfy $(j^\infty \phi)^* \theta = 0$ for all sections ϕ with compact support. This ideal is generated by the contact one-forms, which in local coordinates assume the form $\theta_j^a = du_j^a - u_{i,j}^a dx^i$.

Now let C_0 denote the set of contact one-forms of *order zero*. Contact one-forms of order zero satisfy $(j^1 \phi)^*(\theta) = 0$ and, in local coordinates, they assume the form $\theta^a = du^a - u_i^a dx^i$. Notice that both C_0 and $\Omega^{n,1} = \Omega^{n,1}(J^\infty E)$ are modules over Loc_E . Let $\Omega_0^{n,1}$ denote the subspace of $\Omega^{n,1}$ which is locally generated by the forms $\{(\theta^a \wedge d^n x)\}$ over Loc_E . Let ν denote a volume element on M and notice that in local coordinates ν takes the form $\nu = f d^n x = f dx^1 \wedge dx^2 \wedge \dots \wedge dx^n$ for some function $f : U \rightarrow \mathbf{R}$ and U is a subset of M on which the x^i 's are defined.

Define the operator D_i (total derivative) by $D_i = \frac{\partial}{\partial x^i} + u_{i,j}^a \frac{\partial}{\partial u_j^a}$ (recall we assume the summation convention, i.e., the sum is over all a and multi-index J). For $I = \{i_1 i_2 \dots i_k\}$ where $k > 0$, D_I is defined by $D_I = D_{i_1} \circ D_{i_2} \circ \dots \circ D_{i_k}$. If I is empty $I = \{\}$ then D_I is just multiplication by 1. We also define $(-D)_I = (-1)^{|I|} D_I$. Recall that the Euler-Lagrange operator maps $\Omega^{n,0}(J^\infty E)$ into $\Omega_0^{n,1}(J^\infty E)$ and is defined by

$$\mathbf{E}(P\nu) = \mathbf{E}_a(P)(\theta^a \wedge \nu)$$

where $P \in Loc_E, \nu$ is a volume form on the base manifold M , and the components $\mathbf{E}_a(P)$ are given by

$$\mathbf{E}_a(P) = (-D)_I \left(\frac{\partial P}{\partial u_I^a} \right).$$

For simplicity of notation we may use $\mathbf{E}(P)$ for $\mathbf{E}(P\nu)$. We will also use \tilde{D}_i for $\frac{\partial}{\partial \tilde{x}^i} + \tilde{u}_{iJ}^a \frac{\partial}{\partial \tilde{u}_J^a}$ and $\tilde{\mathbf{E}}_a(P)$ for $(-\tilde{D})_I \left(\frac{\partial P}{\partial \tilde{u}_I^a} \right)$ so that $\mathbf{E}(P) = \tilde{\mathbf{E}}_a(P)(\tilde{\theta}^a \wedge \nu)$ in the $(\tilde{x}^\mu, \tilde{u}^a)$ coordinate system.

Now let \mathcal{F} be the space of functionals where $\mathcal{P} \in \mathcal{F}$ iff $\mathcal{P} = \int_M P\nu$ for some $P \in Loc_E^0$. Let \mathcal{D} be a differential operator which has components $\mathcal{D}^{ab} = \omega^{abI} D_I$ where $\omega^{abI} \in Loc_E$ and D_I is a combination of the total derivatives as determined by the multi-index I , i.e., D_I is a composite of the form $D_{i_1} \circ D_{i_2} \circ \cdots \circ D_{i_k}$. Define a Poisson bracket on the space of local functionals \mathcal{F} by

$$\{\mathcal{P}, \mathcal{Q}\}(\phi) = \int_M [\mathbf{E}(P)\mathcal{D}(\mathbf{E}(Q)) \circ j\phi]\nu,$$

where $\phi \in \Gamma E$, ν is a volume form on M , $\mathcal{P} = \int_M P\nu$, $\mathcal{Q} = \int_M Q\nu$, and $P, Q \in Loc_E^0$. In this expression one may represent \mathcal{D} as a matrix differential operator and \mathbf{E} as a row/column vector (as appropriate) consisting of the components \mathbf{E}_a . We assume that \mathcal{D} is *Hamiltonian* so that we have a genuine Poisson bracket that is antisymmetric and satisfies the Jacobi identity (e.g. see [8]). Using local coordinates (x^μ, u_I^a) on $J^\infty E$, observe that for $\phi \in \Gamma E$ such that the support of ϕ lies in the domain Ω of some chart x of M , one has

$$\{\mathcal{P}, \mathcal{Q}\}(\phi) = \int_{x(\Omega)} ([\mathbf{E}_a(P)\mathcal{D}^{ab}(\mathbf{E}_b(Q))] \circ j\phi \circ x^{-1})(x^{-1})^*(\nu)$$

where x^{-1} is the inverse of $x = (x^\mu)$.

The functions P and Q in our definition of the Poisson bracket (of local functionals) are representatives of \mathcal{P} and \mathcal{Q} respectively, since generally these are not unique. In fact $\mathcal{F} \simeq H_c^n(J^\infty E)$, where $H_c^n(J^\infty E) = \Omega_c^{n,0}(J^\infty E)/(\text{imd}_H \cap \Omega_c^{n,0}(J^\infty E))$ and imd_H is the image of the differential

d_H defined by $d_H = dx^i D_i$. We refer the reader to [1] for more details and for the notation. The interested reader may also consult [2] and [4] for more on the de Rham complex and its cohomology.

Let $\psi : E \rightarrow E$ be an automorphism, sending fibers to fibers, and let $\psi_M : M \rightarrow M$ be the induced diffeomorphism of M . Notice that ψ induces an automorphism $j\psi : J^\infty E \rightarrow J^\infty E$ where

$$(j\psi)((j^\infty \phi)(p)) = j(\psi \circ \phi \circ \psi_M^{-1})(\psi_M(p)),$$

for all $\phi \in \Gamma E$ and all p in the domain of ϕ . In these coordinates the independent variables transform via $\tilde{x}^\mu = \psi_M^\mu(x^\nu)$. Local coordinate representatives of ψ_M and $j\psi$ may be described in terms of charts (Ω, x) and $(\tilde{\Omega}, \tilde{x})$ of M , and induced charts $((\pi^\infty)^{-1}(\Omega), (x^\mu, u_I^a))$ and $((\pi^\infty)^{-1}(\tilde{\Omega}), (\tilde{x}^\mu, \tilde{u}_I^a))$ of $J^\infty E$.

Observe that the total derivatives satisfy

$$D_i(F \circ j\psi) = ((\tilde{D}_j F) \circ j\psi) D_i \psi_M^j \quad (2.1)$$

where $\tilde{x}^j = \psi_M^j(x)$.

Example Let $M = \mathbf{R}^2, E = \mathbf{R}^2 \times \mathbf{R}^1$ and consider the transformation ψ defined by $\tilde{u} = xu + 3yu^2, \tilde{x} = x \cos \theta + y \sin \theta, \tilde{y} = -x \sin \theta + y \cos \theta$. Now let $F = \tilde{u}_{\tilde{x}}$ then $F \circ j\psi = (xu_x + u + 6yuu_x) \cos \theta + (xu_y + 3u^2 + 6yuu_y) \sin \theta$ so that $D_x(F \circ j\psi) = (xu_{xx} + 2u_x + 6yu_x^2 + 6yuu_{xx}) \cos \theta + (xu_{yx} + u_y + 6uu_x + 6yu_xu_y + 6yuu_{yx}) \sin \theta$. On the other hand the right-hand side of equation (2.1) yields $[\tilde{u}_{\tilde{x}\tilde{x}} \cos \theta + \tilde{u}_{\tilde{x}\tilde{y}}(-\sin \theta)] \circ j\psi$ which when evaluated and simplified yields the same expression as above.

Remark For simplicity we may skip writing the tilde's, so in the above example one simply writes $F = u_x$ instead of $F = \tilde{u}_{\tilde{x}}, \dots$ etc.

3 Canonical transformations of the Poisson structure

Let $L : J^\infty E \rightarrow \mathbf{R}$ be a Lagrangian in Loc_E (generally we will assume that any element of Loc_E is a Lagrangian). Let $\tilde{L} = L \circ (x^\mu, u_I^a)^{-1}$ and let

$\tilde{L} = L \circ (\tilde{x}^\mu, \tilde{u}_I^a)^{-1}$. Then, in local coordinates, \tilde{L} is related to \hat{L} by the equation

$$(\tilde{L} \circ j\bar{\psi})\det(J) = \hat{L},$$

where $j\bar{\psi} = (\tilde{x}^\nu, \tilde{u}_K^b) \circ (x^\mu, u_I^a)^{-1}$ and J is the Jacobian matrix of the transformation $\psi_M = \tilde{x}^\nu \circ (x^\mu)^{-1}$. With abuse of notation we may assume coordinates and charts are the same and write $\tilde{x}^\nu = \psi_M(x^\mu)$. For simplicity, we have also assumed that ψ_M is orientation-preserving. In this case the functional

$$\tilde{\mathcal{L}} = \int_{\tilde{\Omega}} \tilde{L} d^n \tilde{x}$$

is the transformed form of the functional

$$\hat{\mathcal{L}} = \int_{\Omega} \hat{L} d^n x$$

where \hat{L} and \tilde{L} are related as above, Ω is the domain of integration and $\tilde{\Omega}$ is the transformed domain under $j\bar{\psi}$ (see [8] pp.249-250). Notice that both of these are local coordinate expressions of the equation $\mathcal{L} = \int_M L\nu$, for appropriately restricted charts. Now suppose that ψ is an automorphism of E , $j\psi$ its induced automorphism on $J^\infty E$, and ψ_M its induced (orientation-preserving) diffeomorphism on M . Also suppose that \hat{L} and \tilde{L} are two Lagrangians related by the equation $(\tilde{L} \circ j\psi)\det(\psi_M) = \hat{L}$. We have:

Lemma 3.0.1 *Let P be a Lagrangian as above, then*

$$\mathbf{E}_a((P \circ j\psi)\det(\psi_M)) = \det(\psi_M) \frac{\partial \psi_E^c}{\partial u^a}(\tilde{\mathbf{E}}_c(P) \circ j\psi). \quad (3.2)$$

Proof First notice that $\mathbf{E}_{u^a}(\hat{L}) = \det(\psi_M) \frac{\partial \psi_E^c}{\partial u^a}(\mathbf{E}_{\tilde{u}^c}(\tilde{L}) \circ j\psi)$ (see [8] pp.250). But $(\tilde{L} \circ j\psi)\det(\psi_M) = \hat{L}$. The identity 3.2 follows by letting $P = \tilde{L}$. Notice that this is justified since \tilde{L} is arbitrary in the sense that given any L' there exists an \hat{L} derived from a Lagrangian L as above such that $(L' \circ j\psi)\det(\psi_M) = \hat{L}$ since $j\psi$ is an automorphism. ■

Let $\hat{\psi}$ denote the mapping representing the induced action of the automorphism on sections of E , i.e., $\hat{\psi} : \Gamma E \rightarrow \Gamma E$ where $\hat{\psi}(\phi) = \psi \circ \phi \circ \psi_M^{-1}$ and ϕ is a section of E . This induces a mapping on the space of local functionals given by

$$\begin{aligned}
\Psi(\mathcal{P}) = (\mathcal{P} \circ \hat{\psi})(\phi) &= \mathcal{P}(\psi \circ \phi \circ \psi_M^{-1}) \\
&= \int_M [P \circ j(\psi \circ \phi \circ \psi_M^{-1})] \nu \\
&= \int_M [P \circ j\psi \circ j\phi \circ \psi_M^{-1}] \nu \\
&= \int_M [P \circ j\psi \circ j\phi] (\det \psi_M) \nu,
\end{aligned}$$

where

$$\mathcal{P}(\phi) = \int_M (P \circ j\phi) \nu,$$

and ϕ is a section of E .

Now we find conditions on those automorphisms of the space of local functionals under which the Poisson structure is preserved for a few special cases of the bundle E and differential operators \mathcal{D} .

3.1 Case I: $\dim(M)=1$, $\dim(E)=2$; $\mathcal{D} = \omega D_x$

We begin with the case where M is 1-dimensional, E is 2-dimensional and consider first order differential operators of the form $\mathcal{D} = \omega D_x$ with $\omega \in Loc_E$. For simplicity we will skip the tilde's in our notation. Recall that $\{\mathcal{P}, \mathcal{Q}\} = \int_M \mathcal{D}(\mathbf{E}(Q)) \mathbf{E}(P) \nu$, and hence

$$\{\mathcal{P}, \mathcal{Q}\}(\phi) = \int_M [\mathcal{D}(\mathbf{E}(Q)) \mathbf{E}(P) \circ j\phi] \nu. \quad (3.3)$$

In this case the Poisson bracket is preserved: $\{\Psi(\mathcal{P}), \Psi(\mathcal{Q})\}(\phi) = \Psi(\{\mathcal{P}, \mathcal{Q}\})(\phi)$ if and only if

$$\begin{aligned}
&\int_M ([\omega D_x \mathbf{E}((Q \circ j\psi) \det \psi_M) \mathbf{E}((P \circ j\psi) \det \psi_M)] \circ j\phi) \nu = \\
&\int_M ([(\omega D_x \mathbf{E}(Q) \mathbf{E}(P)) \circ j\psi \circ j\phi] \det \psi_M) \nu,
\end{aligned}$$

but since this latter equation holds for all sections ϕ of E it is equivalent to

$$\omega D_x \mathbf{E}((Q \circ j\psi) \det \psi_M) \mathbf{E}((P \circ j\psi) \det \psi_M) = [(\omega D_x \mathbf{E}(Q) \mathbf{E}(P)) \circ j\psi] \det \psi_M$$

up to a *divergence*. Using Lemma 3.0.1 this is equivalent to

$$\begin{aligned} \omega D_x (\det \psi_M \frac{\partial \psi_E}{\partial u} \mathbf{E}(Q) \circ j\psi) \det \psi_M \frac{\partial \psi_E}{\partial u} (\mathbf{E}(P) \circ j\psi) = \\ (\omega \circ j\psi) (D_x (\mathbf{E}(Q)) \circ j\psi) (\mathbf{E}(P) \circ j\psi) \det \psi_M \end{aligned}$$

or

$$\begin{aligned} \omega [D_x (\det \psi_M \frac{\partial \psi_E}{\partial u}) (\mathbf{E}(Q) \circ j\psi) + \det \psi_M \frac{\partial \psi_E}{\partial u} (D_x (\mathbf{E}(Q)) \circ j\psi) D_x \psi_M] \\ \frac{\partial \psi_E}{\partial u} (\mathbf{E}(P) \circ j\psi) = (\omega \circ j\psi) (D_x (\mathbf{E}(Q)) \circ j\psi) (\mathbf{E}(P) \circ j\psi) \end{aligned}$$

up to a divergence and where we have used equation (2.1). (Notice that $\det \psi_M$ cancelled from both sides of the equation since it is nonzero.) Finally since the last equation is true for all P and Q we must have $D_x (\det \psi_M \frac{\partial \psi_E}{\partial u}) = 0$ and $\omega \circ j\psi = \omega (\det \psi_M)^2 (\frac{\partial \psi_E}{\partial u})^2$. (One may factor out $(D_x (\mathbf{E}(Q)) \circ j\psi)$ over appropriate domains from the left-hand side of the equation to verify that we must have $D_x (\det \psi_M \frac{\partial \psi_E}{\partial u}) = 0$ since Q is arbitrary, otherwise the equation may not hold for all P and Q . Also notice that by the arbitrariness of P and Q the divergence term must be zero.) Observe that the first condition implies that $\det \psi_M \frac{\partial \psi_E}{\partial u}$ is a constant, while the second condition was simplified using $D_x \psi_M = \det \psi_M$ since M is one-dimensional and so are its fibers.

Theorem 3.1.1 *Let $E \rightarrow M$ be a vector bundle with $\dim(M)=1$, $\dim(E)=2$ and let the Poisson bracket $\{\cdot, \cdot\}$ be defined by $\mathcal{D} = \omega D_x$. Then the induced transformation Ψ on the space of local functionals is canonical, i.e., $\{\Psi(\mathcal{P}), \Psi(\mathcal{Q})\} = \Psi(\{\mathcal{P}, \mathcal{Q}\})$ for all $\mathcal{P}, \mathcal{Q} \in \mathcal{F}$ if and only if*

- (i) $D_x (\det \psi_M \frac{\partial \psi_E}{\partial u}) = 0$, and
- (ii) $\omega \circ j\psi = \omega (\det \psi_M)^2 (\frac{\partial \psi_E}{\partial u})^2$.

3.2 Case II: $\dim(M)=n$, $\dim(E)=m+n$; $\mathcal{D} = \omega^{abi} D_i$

Suppose that M is n -dimensional with fibers of dimension m , and consider first order differential operators of the form $\mathcal{D} = \omega^{abi} D_i$. Preserving the Poisson bracket: $\{\mathcal{P} \circ \hat{\psi}, \mathcal{Q} \circ \hat{\psi}\}(\phi) = (\{\mathcal{P}, \mathcal{Q}\} \circ \hat{\psi})(\phi)$ is equivalent to

$$\int_M ([\omega^{abi} D_i \mathbf{E}_b((Q \circ j\psi) \det \psi_M) \mathbf{E}_a((P \circ j\psi) \det \psi_M)] \circ j\phi) \nu =$$

$$\int_M ([(\omega^{abi} D_i \mathbf{E}_b(Q) \mathbf{E}_a(P)) \circ j\psi \circ j\phi] \det \psi_M) \nu,$$

but since this latter equation holds for all sections ϕ of E it is equivalent to

$$\omega^{abi} D_i \mathbf{E}_b((Q \circ j\psi) \det \psi_M) \mathbf{E}_a((P \circ j\psi) \det \psi_M) = [(\omega^{abi} D_i \mathbf{E}_b(Q) \mathbf{E}_a(P)) \circ j\psi] \det \psi_M$$

up to a *divergence*. Using Lemma 3.0.1 this is equivalent to

$$\omega^{abi} D_i (\det \psi_M \frac{\partial \psi_E^d}{\partial u^b} \mathbf{E}_d(Q) \circ j\psi) \det \psi_M \frac{\partial \psi_E^c}{\partial u^a} (\mathbf{E}_c(P) \circ j\psi) =$$

$$(\omega^{abi} \circ j\psi) (D_i (\mathbf{E}_b(Q)) \circ j\psi) (\mathbf{E}_a(P) \circ j\psi) \det \psi_M$$

or

$$\omega^{abi} [D_i (\det \psi_M \frac{\partial \psi_E^d}{\partial u^b}) (\mathbf{E}_d(Q) \circ j\psi) + \det \psi_M \frac{\partial \psi_E^d}{\partial u^b} (D_j \mathbf{E}_d(Q) \circ j\psi) D_i \psi_M^j]$$

$$\frac{\partial \psi_E^c}{\partial u^a} (\mathbf{E}_c(P) \circ j\psi) = (\omega^{abi} \circ j\psi) (D_i (\mathbf{E}_b(Q)) \circ j\psi) (\mathbf{E}_a(P) \circ j\psi)$$

up to a divergence and where we have used equation (2.1). Finally this can be rewritten as

$$\omega^{cdj} [D_j (\det \psi_M \frac{\partial \psi_E^b}{\partial u^d}) (\mathbf{E}_b(Q) \circ j\psi) + \det \psi_M \frac{\partial \psi_E^b}{\partial u^d} (D_i (\mathbf{E}_b(Q)) \circ j\psi) D_j \psi_M^i]$$

$$\frac{\partial \psi_E^a}{\partial u^c} (\mathbf{E}_a(P) \circ j\psi) = (\omega^{abi} \circ j\psi) (D_i (\mathbf{E}_b(Q)) \circ j\psi) (\mathbf{E}_a(P) \circ j\psi)$$

up to a divergence. As before for this to hold for all P and Q we must have $D_j (\det \psi_M \frac{\partial \psi_E^b}{\partial u^d}) = 0$ for all d, j satisfying $\omega^{cdj} \neq 0$ for some c , and

$$\omega^{abi} \circ j\psi = \det \psi_M \omega^{cdj} \frac{\partial \psi_E^a}{\partial u^c} \frac{\partial \psi_E^b}{\partial u^d} D_j \psi_M^i.$$

Theorem 3.2.1 *Let $E \rightarrow M$ be a vector bundle with $\dim(M)=n$, $\dim(E)=n+m$, and let the Poisson bracket $\{\cdot, \cdot\}$ be defined by $\mathcal{D} = \omega^{abi} D_i$. Then the induced transformation Ψ on the space of local functionals is canonical, i.e., $\{\Psi(\mathcal{P}), \Psi(\mathcal{Q})\} = \Psi(\{\mathcal{P}, \mathcal{Q}\})$ for all $\mathcal{P}, \mathcal{Q} \in \mathcal{F}$ if and only if*

$$(i) \ \omega^{cdj} \neq 0 \text{ for some } c \Rightarrow D_j(\det\psi_M \frac{\partial\psi_E^b}{\partial u^d}) = 0 \text{ for all } b, \text{ and}$$

$$(ii) \ \omega^{abi} \circ j\psi = \det\psi_M \omega^{cdj} \frac{\partial\psi_E^a}{\partial u^c} \frac{\partial\psi_E^b}{\partial u^d} D_j \psi_M^i.$$

3.3 Case III: $\dim(M)=1$, $\dim(E)=2$; $\mathcal{D} = \omega^I D_I$

Lastly we consider the case where we have a one-dimensional manifold M with one-dimensional fibers and n -th order differential operators of the form $\mathcal{D} = \omega^I D_I = \omega^n D_n + \omega^{n-1} D_{n-1} + \dots + \omega^1 D_1 + \omega^0$. For simplicity we assume that I is a *number* rather than a multi-index since in this case the base manifold is one-dimensional so, for example, D_n means $(D_x)^n$ and $D_I = (D_x)^I$. Following similar analysis as before and skipping a few steps the Poisson bracket is preserved if and only if

$$[\omega^0(\det\psi_M \frac{\partial\psi_E}{\partial u} \mathbf{E}(Q) \circ j\psi) + \omega^I D_I(\det\psi_M \frac{\partial\psi_E}{\partial u} \mathbf{E}(Q) \circ j\psi)] \det\psi_M$$

$$\frac{\partial\psi_E}{\partial u} (\mathbf{E}(P) \circ j\psi) = [(\omega^0 \mathbf{E}(Q) + \omega^I D_I \mathbf{E}(Q)) \mathbf{E}(P)] \circ j\psi \det\psi_M$$

or

$$[\omega^0(\det\psi_M \frac{\partial\psi_E}{\partial u} \mathbf{E}(Q) \circ j\psi) + \binom{I}{k} \omega^I [D_k(\det\psi_M \frac{\partial\psi_E}{\partial u}) D_{I-k}(\mathbf{E}(Q) \circ j\psi)]$$

$$\frac{\partial\psi_E}{\partial u} (\mathbf{E}(P) \circ j\psi) = [(\omega^0 \mathbf{E}(Q) + \omega^I D_I \mathbf{E}(Q)) \mathbf{E}(P)] \circ j\psi$$

up to a divergence. Now observe that using equation (2.1) we have

$$\begin{aligned} D_I(F \circ j\psi) &= (D_I F \circ j\psi)(\det\psi_M)^I + (D_{I-1} F \circ j\psi)(I, I-1) + \\ &\quad (D_{I-2} F \circ j\psi)(I, I-2) + \dots + (D_1 F \circ j\psi) D_{I-1}(\det\psi_M) \end{aligned}$$

where $D_I = (D_x)^I$ and we use (I, j) for the $\binom{I-1}{j-1}$ permutations of j $\det\psi_M$'s and $I-j$ D_x 's with $\det\psi_M$ in the rightmost slot (this may be derived by induction). For example $(3, 2) = D_x((\det\psi_M)^2) + \det\psi_M D_x(\det\psi_M)$.

This yields the conditions we must have for a canonical transformation:

$$(i) \omega^0 \circ j\psi = \det \psi_M \omega^0 \left(\frac{\partial \psi_E}{\partial u} \right)^2.$$

$$(ii) \omega^I \circ j\psi = \frac{\partial \psi_E}{\partial u} \sum_{J=I}^n \sum_{j=I}^J \binom{J}{j} \omega^J D_{J-j} \left(\det \psi_M \frac{\partial \psi_E}{\partial u} \right) (j, I); I = 1, 2, \dots, n.$$

Notice that we can combine condition (i) with condition (ii) if we let $(0,0)=1$ and $(j,0)=0$ for $j \neq 0$. To summarize

Theorem 3.3.1 *Let $E \rightarrow M$ be a vector bundle with $\dim(M)=1$, $\dim(E)=2$, and let the Poisson bracket $\{\cdot, \cdot\}$ be defined by the n -th order differential operator $\mathcal{D} = \omega^I D_I$. Then the induced transformation Ψ on the space of local functionals is canonical, i.e., $\{\Psi(\mathcal{P}), \Psi(\mathcal{Q})\} = \Psi(\{\mathcal{P}, \mathcal{Q}\})$ for all $\mathcal{P}, \mathcal{Q} \in \mathcal{F}$ if and only if*

$$\omega^I \circ j\psi = \frac{\partial \psi_E}{\partial u} \sum_{J=I}^n \sum_{j=I}^J \binom{J}{j} \omega^J D_{J-j} \left(\det \psi_M \frac{\partial \psi_E}{\partial u} \right) (j, I); I = 0, 1, \dots, n.$$

Remark If $\det \psi_M$ and $\frac{\partial \psi_E}{\partial u}$ are constant then the above condition reduces to: $\omega^I \circ j\psi = \omega^I (\det \psi_M)^{I+1} \left(\frac{\partial \psi_E}{\partial u} \right)^2; I = 0, 1, \dots, n.$

4 Transformation of evolutionary systems and their conservation laws

Recall that an evolutionary system of equations takes the form

$$u_t = \mathcal{D} \delta \mathcal{H}$$

where \mathcal{D} is a Hamiltonian differential operator and \mathcal{H} the Hamiltonian of the system. The variational derivative δ of a local functional $\mathcal{P} = \int_M P \nu$ has the same components as $\mathbf{E}(P)$, the Euler-Lagrange operator applied to P (we use $\mathbf{E}(P)$ rather than $\mathbf{E}(P\nu)$ for simplicity). In the event of a canonical transformation we have $\{\Psi(\mathcal{P}), \Psi(\mathcal{Q})\} = \Psi(\{\mathcal{P}, \mathcal{Q}\})$. Therefore if \mathcal{P} is a conservation law for the evolutionary system $u_t = \mathcal{D} \delta \mathcal{H}$ with Hamiltonian \mathcal{H} , then $\Psi(\mathcal{P})$ is a conservation law for the evolutionary system $u_t = \mathcal{D} \delta \Psi(\mathcal{H})$ with Hamiltonian $\Psi(\mathcal{H})$.

This transforms the evolutionary system and its conservation laws. We illustrate with a few examples. The first example is somewhat trivial but we find it a good point to start as an illustration of our approach. Also recall that the conditions for a canonical transformation are rather restrictive. In the second example, even though the action is orientation-reversing our conclusions will still hold. The same applies for Example 3.

Example 1 Consider the Korteweg-de Vries equation $u_t = u_{xxx} + uu_x$ with the transformation ψ defined by $\tilde{u} = ku, \tilde{x} = \frac{x}{k}, k > 0$. Even though this transformation is somewhat trivial we find it useful to consider in this example. The Hamiltonian and Hamiltonian differential operator associated with the KdV equation can be chosen to be $\mathcal{H} = \int (-\frac{1}{2}u_x^2 + \frac{1}{6}u^3)dx$ and $\mathcal{D} = D_x$ respectively. Now observe that $\det \psi_M \frac{\partial \psi_E}{\partial u} = 1$ is a constant, and $\omega \circ j\psi = \omega(\det \psi_M)^2 (\frac{\partial \psi_E}{\partial u})^2 = 1$. So ψ defines a canonical transformation of the Poisson bracket defined by $\mathcal{D} = D_x$. This transformation transforms our Hamiltonian to $\mathcal{H}' = \int (-\frac{1}{2k^4}u_x^2 + \frac{1}{6k^3}u^3)dx$ and consequently the KdV equation to $u_t = \frac{1}{k^3}u_{xxx} + \frac{1}{k^2}uu_x$.

Notice that the conservation laws $\mathcal{P}_1 = \int_M \frac{1}{2}u^2 dx, \mathcal{P}_2 = \int_M xu + \frac{1}{2}tu^2 dx$, and $\mathcal{M} = \int_M u dx$ are trivially transformed to $\Psi(\mathcal{P}_1) = \int_M \frac{k}{2}u^2 dx, \Psi(\mathcal{P}_2) = \int_M (\frac{xu}{k} + \frac{k}{2}tu^2)dx$, and $\Psi(\mathcal{M}) = \int_M u dx$.

Example 2 Consider the Korteweg-de Vries equation

$$u_t = u_{xxx} + uu_x$$

as above but this time with the transformation defined by $\tilde{u} = x^2u, \tilde{x} = \frac{1}{x}$. Let $M = \mathbf{R}^+$ and $E = M \times \mathbf{R}^2$. In this case $\tilde{u}_{\tilde{x}} = (x^2u_x + 2xu)(-x^2)$ and the new Hamiltonian is given by $\mathcal{H}' = \Psi(\mathcal{H}) = \int (\frac{1}{2}x^6u_x^2 + 2x^5uu_x + 2x^4u^2 - \frac{1}{6}x^4u^3)dx$. So the KdV equation is transformed to $u_t = D_x \mathbf{E}(H')$ where

$H' = \frac{1}{2}x^6u_x^2 + 2x^5uu_x + 2x^4u^2 - \frac{1}{6}x^4u^3$, i.e., we have

$$u_t = -24x^3u - 2x^3u^2 - x^4uu_x - 36x^4u_x - 12x^5u_{xx} - x^6u_{xxx}.$$

The conservation laws $\mathcal{P}_1 = \int_M \frac{1}{2}u^2dx$, $\mathcal{P}_2 = \int_M xu + \frac{1}{2}tu^2dx$, and $\mathcal{M} = \int_M udx$ are transformed to $\Psi(\mathcal{P}_1) = \int_M -\frac{1}{2}x^2u^2dx$, $\Psi(\mathcal{P}_2) = \int_M -\frac{u}{x} + \frac{k}{2}tu^2dx$, and $\Psi(\mathcal{M}) = \int_M udx$.

Example 3 The Boussinesq equation can be written as a system

$$\begin{aligned} u_t &= v_x, \\ v_t &= \frac{8}{3}uu_x + \frac{1}{3}u_{xxx}. \end{aligned}$$

It has Hamiltonian $\mathcal{H} = \int_M (-\frac{1}{6}u_x^2 + \frac{4}{9}u^3 + \frac{1}{2}v^2)dx$ with associated Hamiltonian differential operator

$$\mathcal{D} = \begin{pmatrix} 0 & D_x \\ D_x & 0 \end{pmatrix}.$$

Let $M = \mathbf{R}^+$ and $E = M \times \mathbf{R}^2$. Now consider the transformation $\tilde{u} = x^2u$, $\tilde{v} = x^2v$, $\tilde{x} = \frac{1}{x}$. The Hamiltonian \mathcal{H} is transformed to $\mathcal{H}' = \Psi(\mathcal{H}) = \int_M (-\frac{1}{6}(-2x^3u - x^4u_x)^2 + \frac{4}{9}x^6u^3 + \frac{1}{2}x^4v^2)\frac{dx}{x^2} = \int_M (-\frac{1}{6}(4x^4u^2 + 4x^5uu_x + x^6u_x^2) + \frac{4}{9}x^4u^3 + \frac{1}{2}x^2v^2)dx$. This transforms the system to

$$\begin{aligned} u_t &= -2xv - x^2v_x, \\ v_t &= -8x^3u - \frac{163}{3}x^4u_x - \frac{62}{3}x^5u_{xx} - \frac{16}{3}x^4u^2 - \frac{8}{3}x^4uu_x - 2x^6u_{xxx}. \end{aligned}$$

The Boussinesq equation has the conservation law

$$\mathcal{P} = \int_M (\frac{1}{6}u_{xx}^2 - 2uu_x^2 + \frac{8}{9}u^4 + 2uv^2 - \frac{1}{2}v_x^2)dx.$$

This is transformed to $\Psi(\mathcal{P}) = \int_M (6x^6u^2 + 6x^8u_x^2 + \frac{1}{6}x^{10}u_{xx}^2 + 12x^7uu_x + 2x^8uu_{xx} + 2x^9u_xu_{xx} - 8x^6u^3 + 8x^7u^2u_x + 2x^8uu_x^2 + \frac{8}{9}x^6u^4 + 8x^6uv^2 + 8x^7uvv_x + 2x^8uv_x^2 - 2x^4v^2 - 2x^5vv_x - \frac{1}{2}x^6v_x^2)dx$.

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